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On tropical matrices of small factor rank

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ABSTRACT

The notion of the factor rank of tropical matrices is considered. We construct a linear-time algorithm that either finds a full-rank 3×3 submatrix of a given matrix A or concludes that the factor rank of A is less than 3. We show that there exist matrices of factor rank 4 whose 4×4 submatrices are all rank deficient.

1. Introduction

The *tropical* arithmetic operations on the set $\mathbb{R}$ of real numbers are given by $a \oplus b = \min\{a, b\}$ and $a \otimes b = a + b$, for all $a, b \in \mathbb{R}$, and the set $(\mathbb{R}, \oplus, \otimes)$ is called the *tropical semiring*. We are interested in studying linear algebra over the tropical semiring, and we define the tropical operations with matrices and vectors by the usual formulas, with tropical arithmetic replacing the usual one. We will also say that a vector $v^0 \in \mathbb{R}^m$ is a *tropical linear combination* of vectors $v^1, \ldots, v^n \in \mathbb{R}^m$ if $v^0 = (\lambda_1 \otimes v^1) \oplus \cdots \oplus (\lambda_n \otimes v^n)$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. One of the important notions of tropical linear algebra is that of the rank of a tropical matrix. In contrast with the situation of the classical linear algebra, there are many different important rank functions for tropical matrices. These rank functions come from tropicalizing different classically equivalent notions of rank, and a deep investigation of these rank functions has been carried out in [1,7]. Our paper is devoted to the notion of the factor rank, which is defined in the following way.

Definition 1.1. The *factor rank* of a tropical matrix $A \in \mathbb{R}^{m \times n}$ is the smallest integer $k \geqslant 1$ such that $A = B \otimes C$, for some $B \in \mathbb{R}^{m \times k}$ and $C \in \mathbb{R}^{k \times n}$.

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The factor rank function can be defined in the same way in the case of matrices over an arbitrary semiring [3]. For matrices over the semiring of nonnegative real numbers, the factor rank is known as the nonnegative rank and has various applications, for instance, in quantum mechanics, statistics, demography, and others [4]. In the case of matrices over the binary Boolean semiring, the factor rank also has important applications and is also known as Schein rank [1]. The factor rank of tropical matrices is also important for some problems of combinatorial optimization [2] and is sometimes called Barvinok rank (see [7, Proposition 2.1]).

The factor rank function of tropical matrices has been investigated in [1]. It was shown that the factor rank is always greater than or equal to the tropical, determinantal, and Gondran–Minoux ranks (one can also find in [1] the definitions of these rank functions). In [1] it was also shown that the factor rank of tropical matrices satisfies the rank-sum and rank-product inequalities. The factor rank of a tropical matrix is known to be greater than or equal to its Kapranov rank, see [7].

Tropical matrices whose factor rank is at most 2 have been widely studied, and the following results are now known for this class of matrices. The set of $d \times n$ matrices whose factor rank is 2 is a simplicial complex [6], and the complete description of this complex has been obtained for $d = 3$ in [6]. Tropical matrices with factor rank 2 have been studied from the topological point of view in [8], and the space of $d \times n$ matrices of factor rank two modulo translation and rescaling has been shown to form a (classical) manifold. The integral homology of this manifold has also been computed in [8]. Proposition 6.1 from [7] states that the factor rank of a tropical matrix is at most 2 if and only if the tropical convex hull of its columns is a path, that is, the tropical convex hull of some pair of the columns. Using the notation of tropical linear combinations, we can reformulate Proposition 6.1 from [7] as follows.

Theorem 1.2 [7]. *A tropical matrix A has factor rank at most 2 if there are indexes i and j such that every row of A is a linear combination of the rows indexed by i and j.*

From the computational point of view, tropical matrices with factor rank at most 2 admit a fast verification algorithm. In fact, the problem of recognizing these matrices can be solved in linear time, see [5,7]. It has been proven in [7] that the factor rank of a matrix is at most 2 if and only if all its 3×3 submatrices have the factor rank at most 2. In our paper, we continue this study and provide a linear-time algorithm that decides whether the factor rank of a given matrix is at most 2, and, if it is not, finds a 3×3 submatrix with factor rank 3. Our algorithm is also related to the interesting problem, arisen from combinatorial optimization [2], of recognizing tropical matrices with bounded factor rank.

For $r \leqslant 3$, every matrix with factor rank r has an $r \times r$ submatrix with factor rank r, see [7]. However, a similar statement for arbitrary r fails to hold in general. Indeed, Proposition 2.2 of [7] gives the counterexample for every integer $r \geqslant 5$. In our paper, we solve the problem in the remaining case $r = 4$, providing an example of a 5×4 matrix of factor rank 4 whose 4×4 submatrices are all of rank at most 3.

The rest of our paper is organized as follows. In Section 2, we construct a linear-time algorithm that either concludes that the factor rank of a given matrix is at most 2 or finds a 3×3 submatrix with factor rank 3. In Section 3, we show that there exist matrices of factor rank 4 with 4×4 submatrices of rank at most 3.

The following notation will be used throughout our paper. An (i, j)th entry of a matrix A will be denoted by a_{ij}, the submatrix of A formed by the rows indexed with $r_1, \ldots, r_p$ will be denoted by $A[r_1, \ldots, r_p]$. We will say that an $n \times n$ tropical matrix B is *full-rank* if the factor rank of B equals n. Otherwise, we will say that B is *rank deficient*.

2. A linear-time algorithm for finding a 3 × 3 submatrix of full factor rank

In this section, we construct a linear-time algorithm which either finds a full-rank 3×3 submatrix of a given matrix A or concludes that the factor rank of A is less than 3. We need the following lemma.

Lemma 2.1. *Assume that a vector $v^0 \in \mathbb{R}^m$ is a tropical linear combination of vectors $v^1, \ldots, v^n \in \mathbb{R}^m$, and set*

$$\mu_i = \max_{j=1}^{m}\{v_j^0 - v_j^i\} \tag{1}$$

for every $i \in \{1, \dots, n\}$. Then we have $v^0 = (\mu_1 \otimes v^1) \oplus \cdots \oplus (\mu_n \otimes v^n)$.

Proof. By the assumption of the lemma,

$$v^0 = \bigoplus_{i=1}^{n} \lambda_i \otimes v^i \tag{2}$$

for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. From (1) it follows that adding $\mu_i \otimes v^i$ does not change the left-hand side of (2). Further, if $\lambda_i < \mu_i$ for some i, then from (1) it follows that $\lambda_i + v_j^i < v_j^0$ for some $j \in \{1, \dots, m\}$. A contradiction with (2) shows that $\lambda_i \geqslant \mu_i$ for every i. Now we can add $(\mu_1 \otimes v^1) \oplus \cdots \oplus (\mu_n \otimes v^n)$ to both sides of (2) and obtain the equality we need. □

Now let us describe an algorithm that either concludes that the factor rank of a given matrix A is at most 2 or finds a triple of indexes (u, v, w) such that the matrix $A[u, v, w]$ has factor rank 3.

Algorithm 2.2. Given a matrix $A \in \mathbb{R}^{n\times m}$ with rows indexed by $r_1, \dots, r_n$.

Step 1. If one of the first three rows (denote its index by r_t) is a tropical linear combination of other two, then return the result of the recursive application of Algorithm 2.2 on the matrix $A[r_1, \dots, r_{t-1}, r_{t+1}, \dots, r_n]$.

Step 2. If the assumption of Step 1 fails to hold, then check whether $n < 3$. If so, then conclude that A has factor rank at most 2; otherwise, return (r_1, r_2, r_3).

Theorem 2.3. *Algorithm 2.2 halts after performing at most $O(mn)$ arithmetic operations. It either concludes that the factor rank of A is at most 2 or returns a triple of indexes (u, v, w) for which the matrix $A[u, v, w]$ has factor rank 3.*

Proof. We can check the assumption of Step 1 in time $O(m)$ by using Lemma 2.1. By the construction of the algorithm, the number of possible recursive calls is at most n. Finally, the computations of Step 2 are easy and always produce an output. This proves the first assertion of theorem.

To prove the second one, we use Theorem 1.2. Then, under the assumption of Step 1, the matrices A and $A[r_1, \dots, r_{t-1}, r_{t+1}, \dots, r_n]$ have simultaneously factor rank either greater than 2 or at most 2. Again, Theorem 1.2 shows that output produced by Step 2 is correct. □

Now we can prove the main result of this section.

Theorem 2.4. *Given $A \in \mathbb{R}^{n\times m}$. There exists an algorithm that requires at most $O(mn)$ arithmetic operations and either finds a 3×3 submatrix of full factor rank or concludes that the factor rank of A is less than 3.*

Proof. Apply Algorithm 2.2. If its output is a triple (u, v, w), then we apply Algorithm 2.2 again on the transpose of $A[u, v, w]$ and obtain an output (r, s, t). Now the 3×3 submatrix of A formed with the rows indexed by u, v, w and columns by r, s, t has factor rank 3 by Theorem 2.3. □

3. A matrix with factor rank 4 whose 4 × 4 submatrices are all rank-deficient

In this section, we show that tropical matrices of factor rank r need not have a full-rank $r \times r$ submatrix even if $r = 4$. We present an example of a matrix of factor rank 4 whose 4×4 submatrices are all rank deficient.

Example 3.1. The factor rank of any of the 4×4 submatrices of the matrix

$$\mathcal{K} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 \\ 4 & 4 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

is at most 3.

Proof. First, the factorization

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 \\ 4 & 4 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 4 & 1 & 0 \\ 4 & 4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 4 & 0 & 1 \\ 4 & 4 & 1 & 0 \end{pmatrix}$$

shows that the factor rank of $\mathcal{K}[1, 2, 3, 4]$ is at most 3. Further, for $\alpha \in \{3, 4\}$ we have

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha & 4 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 0 \\ 3 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ \alpha & 4 & 1 & 0 \end{pmatrix},$$

so the submatrices $\mathcal{K}[1, 2, 3, 5]$ and $\mathcal{K}[1, 2, 4, 5]$ are rank deficient. Finally, for $\beta \in \{-1, 1\}$ we have

$$\begin{pmatrix} 0 & 0 & \beta & 0 \\ 3 & 4 & 1 & 0 \\ 4 & 4 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 4 & 0 \\ 4 & 4 & 0 \\ 3 & 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 4 & 4 & 4 \\ 4 & 0 & \beta & 4 \\ 4 & 4 & 1 & 0 \end{pmatrix},$$

so that $\mathcal{K}[1, 3, 4, 5]$ and $\mathcal{K}[2, 3, 4, 5]$ are also rank deficient. □

In order to prove the main result of this section, we also need the following example.

Example 3.2. The factor rank of the matrix

$$C = \begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 3 & 4 & 1 & 0 \end{pmatrix}$$

equals 4.

Proof. Assume the converse. Then there exist matrices $A \in \mathbb{R}^{4\times3}$ and $B \in \mathbb{R}^{3\times4}$ such that $C = A \otimes B$.

The general strategy of the proof is as follows. First, we will identify a few pairs of entries of A and B whose sum is 0, that is, the minimal element of A. This will allow us to obtain a number of inequalities involving other elements of A and B, and to derive a contradiction.

Step 1. The definition of the tropical matrix multiplication shows that

$$\min_{t\in\{1,2,3\}} \{a_{it} + b_{tj}\} = c_{ij} \tag{3}$$

for every $i, j \in \{1, 2, 3, 4\}$. For $p \in \{2, 3, 4\}$, let t_p be the least argument that attains the minimum in (3) with $i = j = p$. In particular, we then have $a_{pt_p} + b_{t_p p} = c_{pp} = 0$.

Step 2. If $t_p = t_q$ for some $p, q \in \{2, 3, 4\}$, then (3) implies $c_{pq} + c_{qp} \leqslant a_{pt_p} + b_{t_p q} + a_{qt_q} + b_{t_q p}$. So from step 1 it follows that $c_{pq} + c_{qp} \leqslant 0$, and then from the definition of C that $p = q$. So we have $\{t_2, t_3, t_4\} = \{1, 2, 3\}$.

Step 3. By step 2, the minimum in (3) with $i = 2, j = 1$ is provided by some t_u, $u \in \{2, 3, 4\}$, that is, $a_{2t_u} + b_{t_u 1} = 0$. From step 1 we have $a_{ut_u} + b_{t_u u} = 0$, and the Eq. (3) implies $a_{2t_u} + b_{t_u u} \geqslant 0$. So we obtain $a_{ut_u} + b_{t_u 1} \leqslant 0$, and (3) then implies $c_{u1} \leqslant 0$. Thus $u = 2$, that is, $a_{2t_2} + b_{t_2 1} = 0$. Step 1 then implies $b_{t_2 1} = b_{t_2 2}$.

Step 4. By step 2, the minimum in (3) with $i = 1, j = 4$ is attained by some t_v, $v \in \{2, 3, 4\}$, that is, $a_{1t_v} + b_{t_v 4} = 0$. From step 1 we have $a_{vt_v} + b_{t_v v} = 0$, and the Eq. (3) implies $a_{vt_v} + b_{t_v 4} \geqslant 0$. So we obtain $a_{1t_v} + b_{t_v v} \leqslant 0$, and (3) then implies $c_{1v} \leqslant 0$. Thus $v = 4$, that is, $a_{1t_4} + b_{t_4 4} = 0$. Step 1 then implies $a_{1t_4} = a_{4t_4}$.

Step 5. The Eq. (3) shows that both $(a_{1t_4} + b_{t_4 1})$ and $(a_{4t_2} + b_{t_2 2})$ are greater than or equal to 4. Steps 4 and 5 then imply $a_{4t_4} + b_{t_4 1} \geqslant 4$ and $a_{4t_2} + b_{t_2 1} \geqslant 4$. Since $c_{41} = 3$, from (3) it thus follows that $a_{4t_3} + b_{t_3 1} = 3$.

Step 6. From (3) it also follows that both $(a_{1t_2} + b_{t_2 1})$ and $(a_{4t_4} + b_{t_4 2})$ are greater than or equal to 4. Again, steps 4 and 5 imply $a_{1t_2} + b_{t_2 2} \geqslant 4$ and $a_{1t_4} + b_{t_4 2} \geqslant 4$. Since $c_{12} = 3$, from (3) it now follows that $a_{1t_3} + b_{t_3 2} = 3$.

Step 7. Finally, the Eq. (3) shows that both $(a_{1t_3} + b_{t_3 1})$ and $(a_{4t_3} + b_{t_3 2})$ are greater than or equal to 4. So we have $a_{1t_3} + b_{t_3 1} + a_{4t_3} + b_{t_3 2} \geqslant 8$. Steps 5 and 6 imply that, however, $a_{1t_3} + b_{t_3 1} + a_{4t_3} + b_{t_3 2} = 6$. The contradiction obtained completes the proof. □

Now we can compute the factor rank of the matrix $\mathcal{K}$.

Lemma 3.3. *The factor rank of the matrix $\mathcal{K}$ from Example 3.1 equals* 4.

Proof. We set

$$D = \begin{pmatrix} 4 & 4 & 4 & 0 & 1 \\ 4 & 0 & 4 & 4 & 4 \\ 1 & 4 & 4 & 4 & 4 \\ 4 & 4 & 0 & 4 & 4 \end{pmatrix}$$

and note that the product $D \otimes \mathcal{K}$ is equal to the matrix C from Example 3.2. Thus we see that the factor rank of C is less than or equal to the factor rank of $\mathcal{K}$. □

Therefore, the matrix $\mathcal{K}$ from Example 3.1 has factor rank 4, and all the 4×4 submatrices of $\mathcal{K}$ have factor ranks at most 3. In contrast with the situation with the case $r \geqslant 5$, in which there exists a matrix of zeros and ones with factor rank r and rank-deficient $r \times r$ submatrices (see Proposition 2.2 from [7]), the matrix $\mathcal{K}$ has rather a complicated structure. In the subclass of tropical matrices with zeros and ones, we failed to find a 5×4 matrix with factor rank 4 and with all the 4×4 submatrices of factor ranks at most 3. However, rather surprisingly, a 5×4 tropical matrix with these properties and elements from $\mathbb{R}$ does exist and has been found. Now let us prove the main result of this section.

Theorem 3.4. *If the factor rank a tropical matrix A equals $r \leqslant 3$, then A contains an $r \times r$ submatrix of full factor rank. For every integer $r > 3$, there are matrices of factor rank r whose $r \times r$ submatrices are all rank deficient, that is, have factor ranks less than r.*

Proof. For $r = 1$, the statement is trivial. If A has factor rank 2, then by the definition of the factor rank, there is a pair of columns that are not equal up to scaling. Thus we can see that in this case, there are indexes i, j, u, v for which $a_{ij} + a_{uv} \neq a_{iv} + a_{uj}$, so we have proven the case $r = 2$. For $r = 3$, the theorem follows from [7, Proposition 6.1] (as well as from our Theorem 2.4), and for $r > 4$ from [7, Proposition 2.2]. Lemma 3.3 shows that the matrix $\mathcal{K}$ from Example 3.1 proves the case $r = 4$. □

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